The Impacts of Initial Perturbations on the Computational Stability of Nonlinear Evolution Equations

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Abstract The impacts of initial perturbations on the computational stability of nonlinear evolution equations for non-conservative difference schemes and non-periodic boundary conditions are studied through theoretical analysis and numerical experiments for the case of one-dimensional equations. The sensitivity of the difference scheme to initial values is further analyzed. The results show that the computational stability primarily depends on the form of the initial values if the difference scheme and boundary conditions are determined. Thus, the computational stability is sensitive to the initial perturbations.

Keywords: nonlinear evolution equation, initial perturbations, computational stability, initial values


1 Introduction

The numerical simulation of climate, numerical weather prediction, and the numerical simulation of ocean currents all depend on obtaining numerical solutions to evolution equations. Thus, it is crucial to ensure that difference schemes can be computed stably for long time spans. For linear evolution equations solved numerically by finite difference methods, computational stability can be analyzed with the Von Neumann method. Provided that the ratio of the time step to the spatial step satisfies certain restrictions (Courant-Friedrichs-Lewy condition), the difference scheme is computationally stable. Otherwise, it is computationally unstable. For a nonlinear evolution equation, however, the situation is fundamentally different. Computational instability often occurs if finite difference methods are used to solve nonlinear evolution equations.

Lilly (1965) first studied the computational stability of nonlinear evolution equations. Many subsequent studies focused on this issue. Zeng and Ji (1981) have systematically studied a simplified case with heat insulation or without dissipation and investigated the generation of nonlinear computational stability (Zeng, 1978; Zeng and Ji, 1981; Ji, 1981a, b). They first constructed the computational stability of an implicit complete square conservative difference scheme. Following these studies, Ji et al. (1998) constructed the computational stability of an explicit complete square conservative difference scheme (Ji and Wang, 1991; Ji et al., 1998; Wang and Ji, 1990, 1994a, b; Wang et al., 1995). Lin et al. (2000) have developed a new method for judging the computational stability of a non-square conservative difference scheme for a nonlinear evolution equation. In another study, Lin et al. (2002) conducted a comparative analysis and confirmed that the computational stability of the nonlinear evolution equation is completely different in essence from that of the linear evolution equation.

In general, the energy in the atmosphere and the ocean is not conserved in the case of medium- and long-term motion. Atmospheric and oceanic motion are essentially nonlinear, and their medium- and long-term motion status is dependent on the basic state (initial field). Any changes in the basic state will cause changes in the atmospheric and oceanic motion.

In the medium- and long-term numerical weather prediction and numerical simulation of ocean currents, a finite difference scheme is used to obtain a numerical solution to nonlinear atmospheric-oceanic dynamic equations. Therefore, it is of practical importance to study the connection between difference schemes for nonlinear evolution equations and the initial value.

This paper examines the impacts of initial perturbations on the computational stability of non-conservative difference schemes under nonperiodic boundary conditions for one-dimensional nonlinear evolution equations. The sensitivity of the difference scheme to the initial value is given particular attention.

2 Computational stability problems of nonlinear evolution equations

It is difficult to study the computational instability of complicated equations. In fact, the prediction equation includes a nonlinear advection term. Thus, the analysis of the simple nonlinear advection equation can represent the nonlinear characteristics of the prediction equation. Therefore, the one-dimensional nonlinear advection equation is studied in this paper.

For the one-dimensional nonlinear advection equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0, \ a \leq x \leq b, \ 0 \leq t \leq T, \qquad (1)$$

$$u(x, 0) = \phi(x),$$

where $u$ is a function of $x$ and $t$, which denote space and
time variable, $a$, $b$, and $T$ are constant.

Consider the following two difference schemes:

Scheme I: Central-Time-Central-Space (CTCS) scheme

$$
\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} + u_c \frac{u_j^{n+1} - u_j^{n-1}}{2\Delta x} = 0 .
$$

(2)

Scheme II: Lax-Wendroff scheme

$$
\frac{u_j^{n+1} - u_j^n}{2\Delta t} + u_c \frac{u_j^{n+1} - u_j^{n-1}}{2\Delta x} - \left( u_j^n \right)^2 \frac{\Delta t}{2} \frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta x^2} = 0 .
$$

(3)

By analyzing the computational stability of the two schemes above, the two following theorems can be derived (Lin et al., 2000):

Theorem I For scheme I (the CTCS scheme) and scheme II (the Lax-Wendroff scheme) for the one-dimensional nonlinear advection equation, the necessary conditions for computational stability are

$$\frac{\partial u}{\partial x} > 0 .$$

Here, a local linearization method is used to analyze the computational stability of the two difference schemes. At any time level, one can choose the maximum of all function values at that level $u_c = \max_x |u(x,t)|$ to replace the convection coefficient $u$. This choice linearizes the nonlinear equation as follows:

$$
\frac{\partial u}{\partial t} + u_c \frac{\partial u}{\partial x} = 0 .
$$

(4)

Thus, the stability of difference schemes for Eq. (4) can be used as a reference for the stability of the nonlinear evolution equation.

For Eq. (4), the CTCS scheme and the Lax-Wendroff scheme are

Scheme III:

$$
\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} + u_c \frac{u_j^{n+1} - u_j^{n-1}}{2\Delta x} = 0 .
$$

(5)

Scheme IV:

$$
\frac{u_j^{n+1} - u_j^n}{2\Delta t} + u_c \frac{u_j^{n+1} - u_j^{n-1}}{2\Delta x} - \left( u_j^n \right)^2 \frac{\Delta t}{2} \frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta x^2} = 0 .
$$

(6)

respectively.

By applying Fourier series methods to the above two difference schemes, the growth rates can be obtained:

$$
G_1 = \pm (1 - r^2 \sin^2 \theta)^{1/2} - ir \sin \theta
$$

and

$$
G_2 = 1 - 2r^2 \sin^2 \theta - ir \sin \theta ,
$$

respectively, where $r = u_c \frac{\Delta t}{\Delta x}$.

Theorem II A necessary and sufficient condition for the computational stability of scheme III and scheme IV is that the norm of the growth rate is less than 1, i.e., $|r| \leq 1$.

3 Numerical experiments

We designed the following numerical experiments to analyze the relationship between the initial conditions and the computational stability of the nonlinear evolution equation.

Numerical experiment I: we selected the solution area $\Sigma = [0,1] \times [0,10]$ and a space step and time step of 0.01 and 0.001, respectively. The initial condition was $u(x,0) = 5.21078 x + a$, where $a$ was set at 5.21078 or 5.21079.

The left boundary condition was:

$$
u(0,t) = a / (1 + t).
$$

The right boundary condition was:

$$
u(N, t) = 3u_{N-1} - 3u_{N-2} + u_{N-3} .
$$

Evidently, $\max_x |u(x,t)| \leq |u_c| \frac{\Delta t}{\Delta x} \leq 0.7 < 1$. Thus, the scheme satisfies a linear CFL (Courant-Friedrichs-Lewy) stability criterion and a nonlinear criterion. From Fig. 1, for the CTCS scheme we find that the scheme is stable if $a = 5.21078$; if $a = 5.21079$, the scheme shows nonlinear instability (see Fig. 2).

Numerical experiment II: for the Lax-Wendroff scheme, the initial condition is $u(x,0) = 5.21078 x + a$, where $a$ was set at $-2.58805$ or $-2.58806$. Obviously, $\max_x |u(x,t)| \leq |u_c| \frac{\Delta t}{\Delta x} \leq 0.3 < 1$. Thus, the scheme satisfies a linear CFL stability criterion and a nonlinear criterion. From Fig. 3, we can see that if $a = 2.58805$, the scheme is stable; if $a = -2.58806$, the scheme exhibits nonlinear instability (see Fig. 4).

Numerical experiment III: we selected the solution area
\[ \sum = [0,10] \times [0,100] \] and a space step and time step of 0.1 and 0.005, respectively.

The left boundary condition was:
\[ u^n_0 = 2u^n_1 - u^n_2. \]

The right boundary condition was:
\[ u^n_N = 2u^n_{N-1} - u^n_{N-2}. \]

For the CTCS scheme, we selected the initial condition
\[ u(x, 0) = x + a, \] where \( a \) was set at 1.9825, 1.9826, -2.8653, or -2.865299. Obviously, \( \left| a \frac{\Delta t}{\Delta x} \right| \leq \left| b \right| \frac{\Delta t}{\Delta x} \leq 0.6 < 1. \)

Thus, the scheme satisfies the linear CFL stability criterion and the nonlinear criterion. From Fig. 5 and Fig. 7, if \( a \) equals 1.9825 and -2.8653, the scheme is stable; if \( a \) equals 1.9826 and -2.865299, the scheme is unstable (see Fig. 6 and Fig. 8). Therefore, these results suggest that if the construction of the scheme and the boundary conditions are determined, the impact of initial perturbations on computational stability is significant. A very small perturbation may produce computational instability.

### 4 Conclusions

Using theoretical analysis and numerical experiments, we investigated the computational stability of the nonlinear evolution equation and obtained the following conclusions:

1. The computational stability of a nonlinear evolution equation depends not only on the construction of the scheme but also on the initial conditions. If the construction of the scheme is determinate, the computational stability depends primarily on the initial value.
(2) Theorem I is only a necessary condition for the computational stability of the nonlinear evolution equation. Therefore, any computationally stable scheme must satisfy this condition, but satisfying this condition does not guarantee computational stability.

(3) If the construction of the scheme and the boundary conditions are determinate, the computational stability of scheme I and scheme II is very sensitive to the initial conditions. Small perturbations may produce computational instability.

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References


