The Nonlinear Response of the Atmosphere to Large-Scale Mechanical and Thermal Forcing

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ABSTRACT

The subject of large-scale mountain waves is reviewed briefly. Existing mountain wave theory based on a linear system is shown to give an inadequate description of the balance of angular momentum. The response of the atmosphere to mechanical forcing in a nonlinear framework is then discussed, using a two-level quasi-geostrophic long-wave spectral model based on spherical coordinates, including diabatic heating, surface friction and mountains. The nonlinear theory shows that there exists a critical mountain height $H$, which is a function of the frictional coefficient as well as the phase difference between the mountain and the surface pressure field. If, and only if, the mountain height is less than this critical value, can the deflection effect of the mountain be neglected and the response regarded as approximately linear. This critical mountain height is only about 1 km. Thus most of the atmospheric response to large-scale mountains must be nonlinear.

In the nonlinear case, as the mountain height is increased the deflection effect becomes more and more important. Therefore, although in the upper atmosphere the mountain wave is intensified, at the surface the pressure perturbation decreases and the zonal surface winds become dominant and approach an asymptotic value.

It is also shown that the combined effect of mechanical and thermal forcing is nonlinear. Despite the fact that the formation of surface pressure systems is mainly a result of thermal forcing, orography affects, to some extent, their intensities and locations. Considering the balance requirement of angular momentum, it is concluded that purely mechanical forcing cannot be the case in the real atmosphere. Although the mountain torque owes its existence to unevenness of the earth's surface, its sign and intensity depend critically upon the relative locations of mechanical and thermal forcing.

1. Introduction

a. Historical review

The responses of the general circulation of the atmosphere to large-scale orographic forcing have been investigated from three different directions: theoretical research, numerical modeling and data analysis. Since the 1940s, mechanically forced stationary waves have been the subject of many theoretical studies, mostly based on linearized systems. Queney (1948) classified one-dimensional mountain waves by three critical wavenumbers, $k_x = N/\bar{u}$, $k_y = \beta/\bar{u}$ and $k_z = (\beta/\bar{u})^{1/2}$. He also showed that, for all wave-type solutions, high surface pressure is located on the windward side of mountains regardless of the scale of the mountains. Charney and Eliassen (1949) found that the original form of the Rossby wave equation used by Queney cannot adequately describe the behavior of planetary waves. By using the Haurnitz-type equation on a $\beta$-plane with a north–south half-wave width of 33° latitude, they were able to represent the mechanically forced planetary waves along 40°N.

Bolin (1950) extended the problem to two dimensions in the horizontal in order to investigate the response of the atmosphere with a homogeneous basic current to a circular mountain. His solution shows that this response is very sensitive to variations in the horizontal scale of the mountain; only mountains of at least the dimension of the Rockies are significant for the generation of planetary waves in the westerlies. Another important phenomenon revealed by his calculations is the splitting effect of mountains on the basic current impinging on them; this is in accord with the finding of Yeh (1950) that, due to the existence of the Tibetan plateau, the westerly jet is split into two branches not only around, but also downstream of, the plateau.

Eliassen and Palm (1954, 1961) introduced the concept of “radiation boundary conditions.” They found that planetary waves excited by surface terrain can convey energy in the vertical direction "in much..."
the same way as gravity waves": the convergence of the wave energy flux is equal to the conversion of motion, defined as the "secondary source of wave kinetic energy from the basic current into the wave energy," in contrast to its primary source on the surface. They also showed that in westerlies, as sensible heat is transferred poleward, wave energy is carried upward, and the poleward transfer of momentum is accompanied by the equatorward transfer of wave energy. This is supported by the research of Shutts (1978), which showed that an important consequence of the upward propagation of wave energy into the stratosphere in the winter season is the poleward transfer of heat. Thus any rigid upper boundary condition imposed in numerical models may cause a severe reduction of the polarward heat flux by stationary waves throughout the troposphere.

Charney and Drazin (1961), Saltzman (1965), Smith (1979) and Dickinson (1980) investigated the same problem for baroclinic conditions. They found that the upward propagation of planetary wave energy is trapped or reflected in regions where the zonal winds are easterly, or strong westerly. Only if the westerly wind is weaker than a critical value $U_c$, can the planetary wave energy propagate upward. According to typical winter data, only waves with wavenumbers 1 and 2, and possibly 3, can propagate upward; shorter waves are evanescent.

b. The linear response of the atmosphere to mechanical forcing

Most of the theories mentioned above are based on linear frameworks. In such systems, the large-scale mountain wave theories for both the barotropic and baroclinic cases can be briefly reviewed as follows.

1) THE BAROTROPIC RESPONSE

Let an overbar denote the zonal average and an asterisk the deviation from the zonal mean. Then for an arbitrary quantity $A$,

$$ A = \bar{A} + A^* .$$

(1.1)

In a barotropic or equivalent barotropic atmosphere, the linearized form of the vorticity equation [specified later in Eq. (2.1)] under the boundary condition (2.6) in a steady state can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial x} = \frac{f_0}{H} \frac{\partial h}{\partial x} - \frac{f_0}{H} \frac{\partial \psi}{\partial x}.$$

(1.2)

Let the wave-like solution of the streamfunction $\psi$ and the forcing function $h$ be

$$ \psi = (\psi_0 \exp(ikx),$$

(1.3)

and define

$$k_R = (\beta/\bar{u})^{1/2}$$

$$k_f = f_0 \alpha/(H \bar{u})$$

$$k_r^2 = \left[(k^2 - k_R^2) + (k_f k_r)^2\right]^{1/2}$$

$$\theta = \tan^{-1} \frac{k_f k}{k_r^2 - k_R^2}$$

Then, the barotropic response of the atmosphere to mechanical forcing can be concisely demonstrated by the variation of the phase angle $\theta$ as

$$\psi = \gamma e^{i\theta} h$$

$$\gamma = f_0 / (H k_r^2) > 0.$$  

(1.4)

This can be summarized as follows:

1) Without friction, i.e., $k_f = 0$

$$\gamma h, \quad k > k_R$$

for short waves

$$\gamma h, \quad k = k_R$$

for resonant waves

$$-\gamma h, \quad k < k_R$$

for planetary waves.

Thus, $\psi$ is in or out of phase with $h$ for short or planetary waves, respectively, as shown schematically in Fig. 1.1a.

2) With friction

$$\theta = \left\{ \begin{array}{ll}
0 < \theta < \pi/2, & k > k_R \\
\pi/2 < \theta < \pi, & k < k_R
\end{array} \right.$$

(1.5)

as shown in Fig. 1.1b. Since $\gamma$ is always limited, there is no longer resonance. The friction effect, according to (1.6), is to shift the short waves westward and the planetary waves eastward. In other words, the effect of friction is to cause the anticyclone to be located always on the windward side of a mountain in the westerlies.

These conclusions agree with those of other authors, e.g., Saltzman (1965) and Hoskins and Karoly (1981), and can be understood physically as follows. The effect of a mountain in the westerlies is to cause an increase in anticyclonic vorticity of the flow on the windward side. Since the absolute vorticity is conserved during horizontal advection, for small scale perturbations in which the $f$-effect is negligible, relative vorticity $\xi$ will decrease to negative as the flow impinges on the mountain. For large-scale perturbations, geostrophic vorticity is dominant, and the reduction of vorticity of the flow thus causes its deflection toward lower latitudes where $f$ is smaller. Therefore, anticyclones over small mountains and cyclones over large mountains can be expected. When
friction is included, at places where troughs or ridges are located, the right-hand side of (1.2) vanishes, and the effect of ascent and/or descent due to surface undulations should be compensated for by the effect of pumping and/or suction in the Ekman frictional layer. The only possible perturbation is therefore to have an anticyclone on the upwind side and a cyclone on the downwind side of a mountain, regardless of the mountain’s scale.

2) THE BAROCLINIC RESPONSE

Following Charney and Drazin (1961), Hirota (1971) and Smith (1973), the diabatic, frictionless version of Eqs. (2.1) and (2.2) in the absence of transfer by transient eddies can be linearized to give

\[ \tilde{u} \frac{\partial}{\partial x} \nabla^2 \psi + \beta \frac{\partial \psi}{\partial z} = f_0 \left( \frac{\partial}{\partial z} - \frac{1}{H} \right) w, \quad (1.7) \]

\[ \tilde{u} \frac{\partial}{\partial x} \left( \frac{\partial \psi}{\partial z} \right) + \frac{N^2}{f_0} w = 0, \quad (1.8) \]

where the inertial variation of density has been added to (2.1) for convenience of discussion. The linearized lower boundary condition is

\[ w = \tilde{u}_0 \frac{\partial h}{\partial x} \text{ at } z = 0. \quad (1.9) \]

By applying the radiation upper boundary condition (Eliassen and Palm, 1954) and postulating

\[ h(x, y) = h_m \cos y \sin k_x \]

\[ \phi(x, y, z) = \text{Re} \{ \phi(z) \cos \alpha y e^{ik_x} \} \],

\[ n^2 = \frac{N^2}{f_0} \left[ \frac{\beta}{\tilde{u} - (k^2 + l^2)} - \frac{1}{4H^2} \right] \quad (1.10) \]

the baroclinic response of the atmospheric motion to large-scale mechanical forcing can be specified by the types of constraints in (1.10).

i) \( n^2 < 0 \)

This corresponds to

\[ k^2 + l^2 > \frac{\beta}{\tilde{u}} - \frac{f_0^2}{(4H^2N^2)}, \]

i.e., a small-scale mountain. The solution is evanescent in the vertical:

\[ \psi(x, y, z) = \left[ \frac{\rho(0)}{\rho(z)} \right]^{1/2} \left[ \frac{N^2}{f_0} \left[ \frac{1}{|n| - (1/2)H} \right] h \exp(-|n|z). \right] \]

(1.11)

Usually, \( \frac{1}{2}H \) is small compared with \(|n|\), and \( \psi \) is in phase with the mountain and decays exponentially upwards. Since the phase difference between the surface vertical motion \( w_0 \) and \( \psi(x, y, 0) \) [or \( \delta p(x, y, 0) \)] is one quarter of a wavelength, there is no upward propagation of wave energy and the wave is trapped. Since there is no vertical shear in \( \psi \), there is no net poleward transfer of entropy either,
ii) \( n^2 > 0 \)

Now, the Charney-Drazin (CD) criterion is satisfied, i.e.,

\[
0 < \tilde{u} < \tilde{u}_c = \beta [k^2 + l^2 + f_0^2 / (4H^2 N^2)]^{-1} \]

(1.12)

and the solution then corresponds to a vertically propagating planetary wave:

\[
\psi(x, y, z) = \left( \frac{\rho(0)}{\rho(z)} \right)^{1/2} \frac{-f_0 h_m}{\beta/\tilde{u} - (k^2 + l^2)} \cos l y \times \left[ \frac{1}{2H} \cos(kx + nz) + n \sin(kx + nz) \right].
\]

(1.13)

The cosine term inside the last pair of brackets results from the inertial variation of density, but in the case of the incompressible Boussinesq approximation, \( Hn \gg 1 \), and this term vanishes. According to (1.13), the inertial effect tends to cause a lower pressure over the mountain; it cannot then result in any net mountain torque. Also, \( \psi(x, y, 0) \) and \( \omega_0 = \tilde{u} \partial \psi / \partial x \) in quadrature, so there is no associated vertical energy flux. Thus the neglect of the inertial variation of density will not cause severe distortion, at least as far as the balance of angular momentum is concerned. The adoption of the Boussinesq approximation in our model is then justified. The propagating planetary-wave solution (1.13) under this approximation can therefore be simplified to

\[
\psi(x, y, z) = -\frac{N h_m}{[\beta/\tilde{u} - (k^2 + l^2)]^{1/2}} \cos l y \sin(kx + nz),
\]

(1.14)

which is also schematically shown in Fig. 1.1c. Now, since the anticyclone is on the upwind side of the mountain, in phase with \( \omega_0 \), westward mountain torque is created and the wave energy flux is upward. Since the phase of the temperature perturbation

\[
\delta \phi(x, y, z) = \frac{f_0}{g} \frac{\partial \psi}{\partial z} = -B h_m \cos l y \cos(kx + nz)
\]

(1.15)

is 90° to the west of the streamfunction \( \psi \), waves are tilted westward and heat is transferred poleward. For steady-state quasi-geostrophic motion, the wave energy flux (Eliassen and Palm, 1961; Simmons, 1979) can be expressed as

\[
\mathcal{E}_p = \left( u^p \delta p^*, v^p \delta p^* \right) = \left( -\rho_z \tilde{u}^* \delta v^*, \frac{f_0}{B} \rho_z \tilde{u}^* \delta \phi^* \right),
\]

(1.16)

and so poleward fluxes of energy and momentum correspond to upward and equatorward fluxes of wave energy, respectively. Note that

\[
\frac{v^p \delta \phi^*}{\mathcal{E}_p} = \frac{kNBh_m^2}{2[\beta/\tilde{u} - (k^2 + l^2)]^{1/2}} \cos^2 l y
\]

(1.17)

is independent of height, and thus the vertical component of the energy flux is proportional to the zonal wind speed \( \tilde{u} \). The specific wave energy, i.e.,

\[
\tilde{E} = \frac{1}{2} \rho(0) \left( \tilde{u}^2 + \tilde{v}^2 + \frac{\beta}{B} \delta \phi^2 \right),
\]

now becomes

\[
\tilde{E} = \frac{\rho(0)}{4} N^2 h_m \left[ \frac{N^2}{\Omega_n} \right] \left( l^2 \sin^2 l y + k^2 \cos^2 l y + \cos^2 l y \right).
\]

(1.18)

and is also independent of height.

The CD criterion can be used to account for the distinct single-wave structure in the stratosphere. Dickinson (1980) modified (1.12) by changing \( \beta \) to the latitude gradient of potential vorticity \( \delta \partial / \delta y \), and showed that the critical values of \( \tilde{u} \) for waves with wavenumber 1, 2 and 3 are, respectively, 75, 43 and 25 m s\(^{-1}\). For typical Northern Hemisphere winters, therefore, only waves with wavenumber 1, and possibly 2, can propagate upward into the stratosphere.

c. Problems raised from linear theories

Despite the success of the above theories in describing some atmospheric phenomena, there exists a limitation to their applications. Generally speaking, this kind of linear theory encounters the following problems if the mountains are high:

1) The assumption of constant basic current \( \tilde{u} \) amounts to a constant energy conversion to the basic current. This is probably feasible if the scale of perturbation is very small. For mechanically forced planetary waves, the kinetic energy of the perturbation is converted from that of the basic current; they are comparable to each other in magnitude. Thus, when the mountains are sufficiently high, the assumption of constant \( \tilde{u} \) becomes questionable.

2) For the term

\[
\tilde{u} = \tilde{u}_0 + \Delta h \frac{\partial \tilde{u}_0}{\partial z} + \frac{\Delta^2 h}{2!} \frac{\partial^2 \tilde{u}_0}{\partial z^2} + \cdots,
\]

only when \( \Delta h \partial \tilde{u}_0 / \partial z \ll \tilde{u}_0 \) can we assume \( \tilde{u}(\partial h / \partial x) = \tilde{u}_0(\partial h / \partial x) \) (Charney and Drazin, 1960). In the real atmosphere, \( \partial \tilde{u} / \partial z \) is about 3 m s\(^{-1}\) km\(^{-1}\). If we let \( \tilde{u}_0 \) be 5 m s\(^{-1}\), then the perturbation part becomes important if the mountain is higher than 1 km, and cannot be neglected.

3) Letting \( u = \tilde{u} + u^o, v = v^o, h = h^o \), the lower boundary condition can then be separated into linear and nonlinear parts respectively, i.e.,

\[
\nabla \cdot \nabla h = \tilde{u} \frac{\partial h}{\partial x} + \partial (\psi^o, h^o) ,
\]

(1.19)
Since the deflection is included in the nonlinear term, the linearization of the boundary condition takes account only of the effect of ascent, and does not allow air to flow around the mountain. Saltzman and Irsch (1972) used the 850 to 700 mb wind data of Buch (1954) and the smoothed topographic heights of Berkofsky and Bertoni (1955) to calculate the w-field. They showed that the forced vertical motion is indeed greatly exaggerated by the linear boundary condition, and in some places the phase is seriously in error. A similar conclusion was also reached by Egger (1976). Their results implied strong cancellation between the linear part and the nonlinear Jacobian term, and suggested that caution is needed when the linear boundary condition is employed.

4) Moreover, by separating the vorticity equation (2.1) into zonal mean and deviation parts, multiplying them by \(-\bar{\Psi}\) and \(-\psi^*\), respectively, and then integrating over the whole mass \(M\) of the atmosphere, it can be seen that the mountain plays a role in converting the zonal kinetic energy \(K^\star\) into eddy kinetic energy \(K^\ast\), i.e.,

\[
\frac{d\bar{K}}{dt} = \frac{1}{2} \int_M (\bar{\Psi}_m - \bar{\Psi}_s) J(\psi_m - \psi_s, h^*) \times dm + \text{others} \\
\frac{dK^*}{dt} = -\frac{1}{2} \int_M (\bar{\Psi}_m - \bar{\Psi}_s) J(\psi_m - \psi_s, h^*) \times dm + \text{others}
\]

Clearly, the energy conversion is brought about by the nonlinear effect term in (1.19). The linear theory may thus distort some aspects of energetics.

5) Also, as in the case of gravity waves, the balance of angular momentum is neglected in the linear theory of large-scale mountain waves. This can be justified if the wave scale is really small. For planetary waves, the balance requirement of angular momentum becomes a very powerful constraint on the fluid because the effect of the rotation of the earth can no longer be ignored. For example, for a flow with a speed of 5 m s\(^{-1}\) being deflected by mechanical forcing from a west–east to a south–north direction, its total energy is conserved. However, its angular momentum is lost completely. This can significantly affect surface flows and rearrange atmospheric wave patterns in order to reach a new balanced state of angular momentum. Therefore, the neglect of the friction term in linear theories may also distort some aspects of atmospheric angular momentum.

In order to investigate the nonlinear response of the atmosphere to large-scale mechanical forcing, a quasi-geostrophic long-wave spectral model is developed in Section 2. In Section 3 the structure and behavior of the mechanically forced planetary waves are presented and discussed. A nonlinear theory based upon energy conservation and angular momentum balance is then developed in Section 4. In Section 5, the theory is further applied by comparing a case with a nonlinear combination of thermal and mechanical forcing with a corresponding linear combination. Angular momentum aspects are discussed in Section 6, and some significant conclusions are drawn in Section 7.

2. The quasi-geostrophic long-wave spectral model

One of the main subjects of our present study is the balance of angular momentum of the atmosphere, for which the \(\beta\)-effect and the latitudinal distribution of the Richardson number \(R_i\) become important. In order to include the \(\beta\)-effect in the model, the long-wave theory developed by White and Green (1982), based on the \(\beta\)-plane, is here extended to spherical coordinates. It is composed of three parts, the zonal mean part, an explicit zonally asymmetric part representing the effect of stationary planetary waves, and a part implicitly expressing the role of the baroclinic waves. The justification for separating the last two parts in the model is, according to White and Green, that "the wavelength of the most unstable baroclinic wave is about a factor of 2 smaller than the resonant wavelength for stationary orographic or diabatic forcing." In addition, the generation and behavior of these two parts are different. The planetary waves are excited by external forcing and propagate upward toward the stratosphere, while shorter waves are created by the baroclinic instability of the atmosphere, and are evanescent in the vertical.

a. The basic equation system

In a quasi-geostrophic equation set, the vorticity and thermodynamic equations are

\[
\frac{D}{Dt} (\zeta + f) = f_0 \frac{\partial w}{\partial z} - \nabla \cdot (\nabla \psi),
\]

\[
\frac{D}{Dt} \left( \frac{\partial \psi}{\partial z} \right) + \frac{N^2}{f_0} w = -\gamma \left( \frac{\partial \psi}{\partial z} - \frac{\partial \psi}{\partial z} \right) - \nabla \cdot \left( V \frac{\partial \psi}{\partial z} \right),
\]

respectively, where the last terms on the right-hand sides of both equations represent the quantity-transfer by transient eddies. The Boussinesq approximation is applied for the sake of simplicity; it was shown earlier that this approximation does not affect the mountain torque. The Newtonian cooling term, rather than an \(a priori\) specified constant function, is used for the diabatic heating because the distribution of heating sources and sinks is not only the cause, but the result, of the general circulation.

The model atmosphere is divided into two equally deep layers as shown in Fig. 2.1, where a rigid top at
the tropopause, $z = H = 10$ km, is assumed. This might result in a smaller poleward transfer of heat, as discussed in Section 1. However, since the basic feature of atmospheric motion in the troposphere is not affected seriously by the existence of the stratosphere (Eady, 1949; Green, 1960), the application of this condition will not significantly affect the discussions below.

Equations (2.1) and (2.2) are written on odd levels and the interface, respectively, i.e.,

$$\frac{D}{Dt_3}(\xi_3 + f) = -\frac{2f_0}{H}w_2 - \nabla \cdot (V_3^* \xi_3), \tag{2.3}$$

$$\frac{D}{Dt_1}(\xi_1 + f) = \frac{2f_0}{H}(w_2 - w_0) - \nabla \cdot (V_1^* \xi_3), \tag{2.4}$$

$$\frac{D}{Dt_5} \psi_s + \frac{HN^2}{2f_0}w_2 = -\gamma(\psi_s - \hat{\psi}_s) - \nabla \cdot (V_5^* \psi_1), \tag{2.5}$$

where $\psi_s = \psi_3 - \psi_1$ represents the thickness of the model atmosphere, and the substantial derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \nabla \cdot \nabla.$$

The sea-level vertical motion $w_0$ is considered to result from the unevenness of the surface and the pumping and/or suction of the Ekman boundary layer, i.e.,

$$w_0 = v_0 \cdot \nabla h + \alpha \nabla^2 \psi_0, \tag{2.6}$$

where

$$\alpha = \left( \frac{K}{2f_0} \right)^{1/2}$$

and $K$ is the eddy viscosity coefficient, taken as $10$ m$^2$ s$^{-1}$ throughout the experiment. Quantities at the surface and top boundaries are linearly extrapolated from their counterparts at levels 1 and 3. By adding and subtracting the two vorticity equations (2.3) and (2.4), eliminating $w_2$ between the thermodynamic equation and the resultant equation from the subtraction, and transforming them into a nondimensional system, we obtain

$$\nabla^2 \frac{\partial \psi}{\partial t} = -\frac{1}{2} [\mathcal{J}(\psi, \nabla^2 \psi) + \mathcal{J}(\psi, \nabla^2 \psi)]$$

$$- \mathcal{J}(\psi_3 - \psi_5, \eta) - 2 \frac{\partial \psi}{\partial \lambda} - R_d \nabla^2 \psi - 2 \nabla^2 \psi$$

$$+ \nabla \cdot (V \psi \psi) + \nabla \cdot (V^* \psi)$$

$$+ R_d (\nabla^2 \psi - 2 \nabla^2 \psi) + \mu^2 \gamma \psi_1 - \hat{\psi}_1$$

$$- \nabla \cdot (V \psi_3 - V \psi_1) \tag{2.7}$$

where $\psi = \psi_3 + \psi_1 = 2\psi_2$, and $\eta$, the quasi-geostrophic potential vorticity (QGPV), is defined as

$$q = \xi + \frac{\mu}{2} \hat{\psi}_1 \tag{2.8}$$

The nondimensional coefficients are abbreviated as

$$\begin{align*}
\mu^2 &= \frac{8f_0^2 a^2}{N^2 H^2}, & R_d &= \frac{f_0 \alpha}{\Omega H} \tag{2.9}
\end{align*}$$

the dimensionless Coriolis parameter

$$f = 2 \sin \varphi = 2x, \tag{2.10}$$

and the dimensionless Laplacian and Jacobian operators are

$$\nabla^2 = a^2 \nabla^2 \tag{2.11}$$

$$\mathcal{J}(A, B) = a^2 \mathcal{J}(A, B), \tag{2.12}$$

respectively. Since in the quasi-geostrophic system the mountain itself does not generate net kinetic energy, the wind velocity at level 1, rather than at the surface, is used in the mountain terms in Eq. (2.7) in order to satisfy the consistency of the difference scheme.

b. The solution of the model

In an adiabatic, frictionless system with a uniform lower surface, quasi-geostrophic potential vorticity is conserved. The transfer theory proposed by Green (1970) can then be used to parameterize the transfer of potential vorticity by transient eddies, i.e.,

$$\begin{align*}
\nabla^2 \psi q_3 &= -K^3 \psi q_3 \\
\nabla \cdot (V \psi q_1) &= -K^3 \nabla \cdot (V \psi q_1) \tag{2.12}
\end{align*}$$
where the dimensionless transfer coefficient $K_i^* = a^{-2} \Omega^{-1} K_i$. The selection of the $K$ values is the key to the parameterization. In our study, the $K$ are treated as functions with their maxima at midlatitudes and vanishing at the poles and equator as in (2.14). The variation of the $K$ in the longitudinal direction is ignored. Thus, the transient eddies directly exert a dissipative effect upon planetary waves. However, the indirect connection between these two parts through the zonal mean flow seems more important for the intensification of planetary waves.

On substituting (2.11) and (8.8) into (2.7), the prognostic equations become

$$
\begin{align*}
\nabla^2 \frac{\partial \psi_M}{\partial t} + (\nabla^2 - \mu^2) \frac{\partial \psi_s}{\partial t} &= \left( \frac{G_1}{G_2} \right) (\psi_M, \psi_s; \eta, \tilde{\psi}, K^\dagger, \tilde{K}^\dagger).
\end{align*}
$$

(2.13)

The functional forms of $G_1$ and $G_2$ are shown in Appendix A.

In order to obtain the numerical solutions, we choose the following spherical harmonic functions for the expansion of dependent variables and forcing functions:

$$
\begin{align*}
\left( \begin{array}{c}
\psi_M \\
\psi_s \\
\eta \\
\end{array} \right) &= \sum_{n} \sum_{m} \left( \begin{array}{c}
\psi_{mn} \\
\psi_{mn} \\
\eta_{mn} \\
\end{array} \right), \\
&= \sum_{n} \sum_{m} \left[ \begin{array}{c}
A_{nm}(t) \\
B_{nm}(t) \\
\end{array} \right] \cos m \lambda \\
&+ \left[ \begin{array}{c}
A_{nm}(t) \\
B_{nm}(t) \\
\end{array} \right] \sin m \lambda P_{nm}(x),
\end{align*}
$$

(2.14)

$$
\left( \begin{array}{c}
K^\dagger \\
\tilde{K}^\dagger \\
\end{array} \right) = \left( \frac{k_3}{k_1} \right) \left( \begin{array}{c}
\tilde{T}_f(x) \\
\tilde{T}_M(x) \\
\end{array} \right),
$$

where the constant $k_1$ corresponds to $K_1 = 4 \times 10^6$ m$^2$ s$^{-1}$, and constant $k_3$ is proportional to $k_1$ by a constant $p$. The value of $p$ is determined by the net zero surface torque constraint

$$
\int_0^1 (\tilde{T}_f + \tilde{T}_M) dx = 0,
$$

(2.15)

because, in the long term, the net angular momentum of the atmosphere is constant. Here, $T_f$ and $T_M$ are the surface frictional torque and mountain torque, respectively, and the overbar represents the zonal mean operator, i.e.,

$$
\bar{A} = \frac{1}{2\pi} \int_0^{2\pi} A(\lambda) d\lambda,
$$

(2.16)

and $P_{nm}(x)$ are the normalized associated Legendre functions determined by the Rodrigues formula

$$
P_{nm}(x) = \left[ \frac{(2n + 1)(n - m)!}{(2n + m)!} \right]^{1/2} \times \frac{(1 - x^2)^{m/2}}{2^n m!} \frac{d^{n+m}}{dx^{n+m}} (x^2 - 1)^n.
$$

(2.17)

In order to trace the connection between the physics and the model results as clearly as possible, we try to avoid dealing with a complicated model, and choose only two modes, $P_0^0$ and $P_1^0$, for the zonal mean, and four modes, $P_2^1$, $P_3^1$, $P_2^2$, and $P_3^2$ for the zonally asymmetric part. The meridional variations of these modes are shown in Fig. 2.2.

Upon substituting (2.14) into (2.13), and using the orthogonality of the associated Legendre functions as well as the trigonometric functions, we obtain, after laborious manipulation, 20 “tendency” equations of the form

$$
\begin{align*}
\frac{dA_{nm}(t)}{dt} &= F_1, \\
\frac{dB_{nm}(t)}{dt} &= F_2, \\
\frac{da_{nm}(t)}{dt} &= F_3, \\
\frac{db_{nm}(t)}{dt} &= F_4,
\end{align*}
$$

(2.18)

and the constraint

$$
p = (-d_1/d_3) + d_1.
$$

(2.19)

The functional forms of $F_1$, $F_2$, $F_3$, $F_4$, $d_1$, $d_2$, and $d_3$ are also given in Appendix A.

Now, Eqs. (2.18) and (2.19) form a closed set with 21 equations and 21 variables, which can be solved numerically. A centered difference scheme is used for all terms except the dissipation term and the term representing vorticity transfer by transient eddies. For these two terms, a time-forward-difference scheme is required (see Appendix B). The temporal increment is fixed as 6 h. All integrations are started from a resting atmosphere and carried to a quasi-steady state until the sequential integrated values of $p$ at the $(j + 1)$th step satisfy the accuracy requirement

$$
|p^{j+1} - p^j| < 10^{-5}.
$$

(2.20)

Results from this state (referred to as the model atmosphere), including different meteorological fields, are diagnosed so as to investigate the roles of orography in atmospheric motions.
3. The mechanically forced planetary wave

Most of the linear theories have been concerned only with steady state ($\partial / \partial t = 0$) solutions to some very special problems of large-scale forcing in constant basic flow without including some other factors. Thus, they are in fact just some refurbished versions of gravity-wave theories. As the permissible upward propagating waves in our study possess a scale comparable with that of the planet itself, those “other factors,” such as the $\beta$-effect, nonlinear advection, atmospheric energetics and angular momentum balance, etc., may become rather important, even critical, in some circumstances. Our present interest is therefore to consider to what extent the linear theory can be accepted and what modification may be expected.

Fig. 2.2. Modes of the associated Legendre functions used in the model: $P_2^0$ and $P_2^2$ for the zonal mean quantities; $P_2^1$, $P_3^1$, $P_3^3$ and $P_3^5$ for the zonally asymmetric quantities.

Fig. 3.1. Composite map of the sectorial and tesseral harmonics of the earth's topography (m) above mean sea level, up to the 4th degree (a) and 8th degree (b). From Figs. 1 and 2 of Sankar-Rao (1965).
on inclusion of these additional factors. As a start, in this section we examine the linear theory using our general circulation model.

The spectral decomposition of the elevation of the earth's surface by Sankar-Rao (1965) exhibits a highly complicated structure of unevenness: for \( m < 10 \), the coefficients \( hA_m \), \( hB_m \) in (2.14) for some large wave-numbers are comparable in magnitude with the maximum coefficient. Hence it is impossible to use a highly truncated model to simulate the real orography. Fortunately, from the above discussions we see that the role of mountains with different scales, at least in the case of mechanical forcing, is so distinctive that for our purposes the use of large-scale mountains alone in our investigation of mechanical forcing can be justified. Figure 3.1 gives composite maps of the sectoral and tesseral harmonics of the earth's orography above mean sea level (m) to the 4th (a), and 8th (b) degree. In our mechanical forcing experiment, an idealized mountain with \( hA_2 = -500 \text{ m/}H \), \( hB_2 = 250 \text{ m/}H \) is put into a zonally symmetric diabatic heating field, where \( H = \frac{\omega H_0}{f} = 7854 \text{ m} \) is a constant height measure. This mountain distribution gives two peaks, at 90°E and 90°W, and two valleys, at 0° and the dateline, as shown in Fig. 3.2. The reference temperature field is specified by setting \( S_{A2,0} = -30 \text{ K/}T \) and \( S_{A4,0} = 7 \text{ K/}T \), where \( T = 2.66\pi\sigma T^2 R^{-1} = 1750 \text{ K} \) is a constant temperature measure. This corresponds to a zonally mean temperature difference of 21 K between the equator and the pole in the steady state, and creates the basic westerly shear in the model.

Experiments show that the circulation is no longer zonally symmetric when these mountain effects are allowed for. The vertical structure of the forced mountain waves along the 40°N latitude circle at

---

**Fig. 3.2.** The assumed orography (m) with \( hA = -500 \text{ m/}H \) and \( hB = 250 \text{ m/}H \).

**Fig. 3.3.** Cross section of the structure of planetary waves forced by large-scale orography along 40°N latitude. Open circles represent trough lines.
which mountain peaks and valleys are located is shown in Fig. 3.3. The surface high pressure is on the western side of mountains and in phase with the forced surface ascent. At the middle level, $\psi_2$ is out of phase with the orography. The temperature perturbation is $\sim 3$ K in amplitude, in agreement with $(2Bh_mT)$ given by (1.15), and lags behind the streamfunction $\psi_2$ by about 30°. Systems are therefore tilted westward with height. Therefore, the entropy flux is poleward and wave energy propagation is upward in the middle level. The whole picture is in good agreement with the linear theory shown in the first section.

Figure 4.2 shows the response of $\Delta P_0$ and $\Delta P_0^s$ to different mountain heights. In contrast, we see that linear theory always overestimates the response, but when the mountain height $h (=2h_m)$ is less than 1 km, this overestimate is small and the approximations of linear theory seem reasonable.

The nonlinear response is characterized by the two-stage structure. Both $\Delta P_0$ and $\Delta P_0^s$ increase, with $\Delta P_0^s > \Delta P_0$ for $h$ up to $\sim 2$ km; above this height, $\Delta P_0^s$ decreases gradually while $\Delta P_0$ increases to approach an asymptotic value. The increase of $\Delta P_0^s$ for small $h$ suggests that although the relative importance of the nonlinear Jacobian in (1.19) is
increasing, the climbing effect due basically to the linear part is still dominant. Its decrease for larger \( h \) implies that the nonlinear Jacobian becomes overwhelming. These responses are summarized in Table 4.1.

b. The theory of the nonlinear response

What we have discussed about the nonlinear response so far is no more than a result from a simple model, and deserves further discussion. With reference to Fig. 4.2, we should like to know the reason for the shape of the curves and also whether they are, in general, applicable to all mechanical forcings. To answer these questions we now investigate the response of the atmosphere to mechanical forcing from the viewpoint of the balance requirements of both energy and angular momentum. Since the geostrophic relation implies that \( \Delta \tilde{P}_0 \) and \( \Delta \tilde{P}_g \) are related to the zonal mean zonal surface wind \( u_0 \) and the surface perturbation wind \( u_0^* \), respectively, we now consider, for convenience, the surface wind instead of the pressure field.

1) THE ENERGY SPHERICAL SURFACE

The equations of gross kinetic and potential energy for the whole atmosphere over the Northern Hemisphere can be readily deduced from (2.3) to (2.5) as being

\[
\frac{dK}{dt} = \frac{2f_0}{H} \int_M w_\psi \, dm - \int_S f_0 \alpha (\nabla \psi_0)^2 \, dS, \tag{4.1}
\]

\[
\frac{dA}{dt} = -\frac{2f_0}{H} \int_M w_\psi \, dm + \int_M \gamma (\psi_\psi - \psi_0^2) \, dm, \tag{4.2}
\]

where \( \nu^2 = 4f_0^2/(N^2H^2) \), and \( S \) and \( M \) are the total surface area and mass of the atmosphere, while

\[
K = \int_M \frac{1}{2} (\nabla \psi)^2 \, dm,
\]

\[
A = \int_M \frac{1}{2} \nu^2 \psi_0^2 \, dm
\]

represent the gross kinetic and potential energy, respectively. Notice that, as expected, the mountain term disappears from the total energy budget in (1.2). The first term on the right-hand side of (4.1) and (4.2) represents the energy conversion from the potential to the kinetic form. In a long term mean, or in a steady state, the total energy of the atmosphere is unchanged, so the generation of available potential energy must be balanced by the dissipation of kinetic energy, i.e.,

\[
\int_M \nu^2 \gamma (\psi_\psi - \psi_0^2) \, dm - \int_S f_0 \alpha (\nabla \psi_0)^2 \, dS = 0. \tag{4.3}
\]

Let us choose the following approximate solutions to the steady-state equation in the \( x-y \) coordinates in the domain \( S = L_x \times L_y \):

\[
\psi = \hat{\psi} \cos \nu y \sin \omega x + \psi^* \cos \omega x \sin \nu y + h^* \sin \omega x \
\]

\[
\psi^* = \sqrt{2} \psi^* \sin \nu y \cos \omega x \tag{4.4}
\]

Then

\[
\hat{u}_0 = -U_0 \sin \nu y,
\]

\[
u_0^* = -\sqrt{2} U_1 \sin \omega x \sin \nu y
\]

\[
u_0^* = -\sqrt{2} V_1 \sin \omega x \sin \nu y
\]

where \( \nu \) is used to represent the phase difference between the pressure (streamfunction) perturbation and orography, and

\[
U_0 = m \hat{\psi}, \quad U_1 = m \psi^*, \quad V_1 = k \psi^*.
\]

Then

\[
\int_S (\nabla \psi_0)^2 \, dS = \int_S [(\hat{u}_0 + \nu_0^*)i + \nu_0^* j]^2 \, dS = \int_S (\hat{u}_0^2 + \nu_0^2 + \psi_0^2) \, dS
\]

\[
= \frac{L_x L_y}{2} (U_0^2 + U_1^2 + V_1^2), \tag{4.5}
\]

where \( m_0 = 2/a, \ Ly = \pi a/2 \), and periodic boundary conditions in the \( x \)-direction have been applied. After introducing the concept of "energy-speed radius" \( R_{en} \) as

\[
R_{en}^2 = \frac{2}{\nu^2} (f_0 \alpha L x L y)^{-1} \int_M (\psi_0 \tilde{\psi} - \psi_0^2 \tilde{\psi}) \, dm
\]

where \( \nu^2 = 4f_0^2/(N^2H^2) \), and \( S \) and \( M \) are the total surface area and mass of the atmosphere, while
and substituting (4.5) and (4.6) into (4.3), we obtain

$$U_0^2 + U_1^2 + V_1^2 = R_{en}^2. \quad (4.7)$$

If we define the "surface wind space" as \{U_1, V_1, U_0\}, the energy conservation relation (4.3) now becomes the equation of a spherical surface (4.7), with energy-speed \(R_{en}\) as its radius in this new space, as shown in Fig. 4.3.

The spherical surface (4.7) is defined as the "energy spherical surface" since its size depends on the energy supply. It can also be understood physically as follows. In any steady-state general circulation model with only surface dissipation of kinetic energy, the amplitude of the surface wind depends solely upon the energy source of the system. The greater the energy supply, the larger the surface wind speed. Any steady solution should have its surface wind located on this energy spherical surface in \{U_1, V_1, U_0\} space. The theoretical maximum speed of both the zonal and the deviation surface winds is \(R_{en}\). If the frictional coefficient \(\alpha\) is changed, then from (4.6) and (4.7) we obtain an interesting result: For a general circulation model with fixed energy supply, any relative decrease of the frictional coefficient \(\alpha\) will result in a half relative increase of the surface wind amplitude, and vice versa, i.e.,

$$\frac{d\alpha}{\alpha} + \frac{2d[|V|]}{|V|} = 0.$$

Since \(U_1\) and \(V_1\) are related to the same streamfunction, therefore, the conservation of energy is represented by the intersection circle of the energy spherical surface and the plane \(U_1 = (m/k)V_1\), i.e.,

$$U_1^2 + V_1^2 + U_0^2 = R_{en}^2 \quad \left\{ \begin{array}{l}
U_1 = (m/k)V_1 \\
\end{array} \right. \quad (4.8)$$

2) THE ANGULAR MOMENTUM PLANE

From the vorticity equation (2.1) and boundary condition (2.6), the following angular momentum equation can be derived:

$$\frac{\partial \bar{\mu}}{\partial t} = \int_0^H \rho \bar{\omega} \bar{\tau} \cos \varphi dz$$

$$+ \int_0^H \rho \bar{v} \bar{\tau} \cos \varphi dz + \bar{\tau}_f + \bar{\tau}_M + \bar{s}. \quad (4.9)$$

Here, \(\bar{\mu} = \int_0^H \rho \bar{\omega} \bar{\tau} \cos \varphi dz\) is the zonal mean westerly relative angular momentum, \(\bar{\tau}_f = -\bar{\tau}_f \cos \varphi\) the surface frictional torque exerted on the atmosphere by the earth, \(\bar{\tau}_s = \int_0^H \rho \bar{v} \bar{T} \cos \varphi dz\) the surface stress exerted on the earth by the atmosphere, \(\bar{\tau}_M = \bar{\tau}_M \cos \varphi\) the mountain torque exerted on the atmosphere, \(\bar{\tau}_M = \int_0^H \rho \bar{v} \bar{\omega} \varphi^2 \cos \varphi dz\) the mountain drag exerted on the atmosphere, and \(\bar{s}\) represents other sources of angular momentum and is assumed to be zero here.

From (4.9) we can obtain the atmospheric angular momentum balance equation:

$$\int_{\varphi_1}^{\varphi_2} \frac{\partial \bar{\mu}}{\partial t} 2\pi a^2 \cos \varphi d \varphi = \int_0^H 2\pi a^2 [\rho \bar{u} \bar{w}] \psi \cos^2 \varphi_1 dz$$

$$- \int_0^H 2\pi a^2 [\rho \bar{v} \bar{w}] \psi \cos^2 \varphi_1 dz$$

$$= \int_{\varphi_1}^{\varphi_2} 2\pi a^2 \bar{\tau}_f \cos \varphi d \varphi + \int_{\varphi_1}^{\varphi_2} 2\pi a^2 \cos \varphi \bar{\tau}_M d \varphi + S. \quad (4.10)$$

If we define the following quantities in the polar cap area northward of latitude \(\varphi_1\):

$$T_{\bar{\tau}_f} = \int_0^H 2\pi a^2 [\rho \bar{u} \bar{w}] \psi \cos^2 \varphi_1 dz,$$

dpoleward angular momentum transfer across latitude \(\varphi_1\) by stationary waves;

$$T_{\bar{\tau}_f} = \int_0^H 2\pi a^2 [\rho \bar{u} \bar{w}] \psi \cos^2 \varphi_1 dz,$$

dpoleward angular momentum transfer across latitude \(\varphi_1\) by transient eddies;

$$T_f = \int_{\varphi_1}^{\varphi_2} 2\pi a^2 \bar{\tau}_f \cos \varphi d \varphi,$$

gross surface frictional torque exerted on the atmosphere by the earth;

$$T_M = \int_{\varphi_1}^{\varphi_2} 2\pi a^2 \bar{\tau}_M \cos \varphi d \varphi,$$

gross mountain torque exerted on the atmosphere;

$$T_m = -\int_{\varphi_1}^{\varphi_2} 2\pi a^2 \frac{\partial \bar{\mu}}{\partial t} \cos \varphi d \varphi,$$

source of angular momentum due to the spindown of the atmosphere; and
\[ S = - \int_{\varphi_1}^{\varphi_2} 2\pi a^2 \cos \varphi \delta \varphi, \]

other sources of angular momentum; then (4.10) becomes

\[ (T_{ys} + T_{\gamma 0}) + (T_T + T_M + T_{sp} + S) = 0. \tag{4.11} \]

Both (4.10) and (4.11) simply state that, in any polar cap area, the total production of angular momentum (the second pair of parentheses in 4.11) should be balanced by the trans-boundary flux of angular momentum. In this study, \( T_{sp} = S = 0 \). Since, in the pure mechanical forcing example, \( T_{\gamma 0} \) is negligible and \( T_M = 0 \), the mountain torque \( T_M \) should be balanced by the frictional torque. If we set \( \varphi_1 = 0 \), Eq. (4.11) then amounts to

\[ \int_S f_0 \rho_0 \nabla \times \mathbf{h}^* a \cos \varphi \, ds = \int_S f_0 \rho_0 \alpha \bar{u}_0 a \cos \varphi \, ds. \tag{4.12} \]

Substituting (4.4) into (4.12), we obtain

\[ U_0 = C \frac{H_1}{\alpha} V_1 \]

\[ C = \frac{3}{8} \sqrt{2} \sin \lambda_0 \tag{4.13} \]

Now, in the surface wind space \( \{ U_1, V_1, U_0 \} \), the requirement of angular momentum balance may be visualized as an "angular momentum plane" passing through the coordinate axis \( U_1 \) with a slope angle \( \theta \) as shown in Fig. 4.3, where

\[ U_0 = \tan \theta V_1 \]

\[ \theta = \tan^{-1} \left( \frac{3}{8} \sqrt{2} \sin \lambda_0 \frac{H_1}{\alpha} \right) \tag{4.14} \]

The response of the surface wind to purely mechanical forcing is therefore represented by the slope of this angular momentum plane, and can be described as follows:

1) When there is no friction, i.e., \( \alpha = 0 \), then \( \lambda_0 \) equals 0 or \( \pi \), the perturbation is either in phase or out of phase with orography, and either the mountain torque or the zonal surface wind vanishes. When friction is permitted, i.e., \( \alpha \neq 0 \), then \( 0 < \lambda_0 < \pi \) and the mechanically forced surface anticyclone is on the western side of the mountain, in accord with the barotropic case shown in Fig. 1.1.

2) When \( \lambda_0 = \pi/2 \), either the mountain torque or the zonal surface wind reaches its maximum. Since the mechanically forced vertical motion and pressure perturbation are now in phase, the maximum upward energy flux is accompanied by the maximum mountain torque and zonal mean zonal surface wind.

3) When there is no mountain, \( \tan \theta = 0 \) and the momentum plane becomes \( U_0 = 0 \), coincident with the coordinate plane \( \{ U_1, V_1 \} \). Since in purely mechanical forcing there is no eddy transfer of angular momentum, the vanishing of the zonal mean zonal surface wind is in accord with the theory developed by Jeffreys (1926) and Eady (1950), which states that in an atmosphere with a smooth lower boundary, the zonal mean zonal surface wind can only be maintained by the eddy transfer of angular momentum.

4) For shallow mountains, \( \theta \) is small and so the perturbation winds predominate. As the mountain height \( H_1 \) increases, the plane becomes steeper and the zonal surface wind becomes dominant.

3) THE CRITICAL MOUNTAIN HEIGHT

Combining the conservation of energy and the requirement of angular momentum balance, the response of surface wind (or pressure) to mechanical forcing can be determined by three geometric surfaces in the \( \{ U_1, V_1, U_0 \} \) space as follows:

\[ U_1^2 + V_1^2 + U_0^2 = R_m^2 \]

\[ U_1 = \frac{m}{k} \frac{V_1}{\alpha} \]

\[ U_0 = \tan \theta V_1 \quad \text{with} \quad \theta = \tan^{-1} \left( \frac{3}{8} \sqrt{2} \sin \lambda_0 \frac{H_1}{\alpha} \right) \tag{4.15} \]

The solution of (4.15) is therefore immediate, i.e.,

\[ U_0 = R_m \left[ 1 + \left( \frac{\alpha}{CH_1} \right)^2 \left( 1 + \frac{m^2}{k^2} \right) \right]^{-1/2} \]

\[ V_1 = R_m \left[ \left( \frac{CH_1}{\alpha} \right)^2 \left( 1 + \frac{m^2}{k^2} \right) \right]^{-1/2} \]

\[ U_1 = R_m \left[ \left( \frac{CH_1}{\alpha} \right)^2 + \left( 1 + \frac{m^2}{k^2} \right) \right]^{-1/2} \tag{4.16} \]

This shows that \( U_0 = 0 \) when \( H_1 = 0 \), and that \( V_1 = 0 \) when \( H_1 \) approaches infinity. For the problem on a planetary scale, it is reasonable to assume \( m^2 \ll k^2 \). If we define the critical mountain height \( H_c \) as

\[ H_c = 2\alpha/C, \tag{4.17} \]

and a dimensionless factor

\[ \lambda = H_c \alpha \tag{4.18} \]

then (4.16) can be well approximated by two different categories:

1) \( 2H_1 \ll H_c \), i.e., \( \lambda_c \ll 1 \)

\[ V_1 = R_m \]

\[ U_0 = R_m \lambda \]

2) \( 2H_1 \gg H_c \), i.e., \( \lambda_c \gg 1 \)

\[ V_1 = R_m \lambda_c^{-1} \]

\[ U_0 = R_m \]
Hence, for small mountains $V_1 > U_0$, while for high mountains $U_0 > V_1$.

Having obtained the relations (4.15)–(4.20) for the case of the nonlinear response of the atmosphere to mechanical forcing, we can now return to Fig. 4.2 to discuss the two-stage structure. The energetics relations (4.1) and (4.2) imply that the supply of mechanical energy to the model atmosphere is mainly determined by the covariance between temperature and vertical motion. As the mountain is inserted into the model and its height is raised gradually, a stationary wave develops. Consequently, the term $\int_M w^* \psi^* \, dm$ gets larger, resulting in the rapid increase in the energy supply. According to (4.19), the rapid increase in $R_m$ causes a rapid increase of both $U_0$ and $V_1$ (with the former increasing to a lesser extent than the latter), as shown schematically in Fig. 4.4a. After the mountain height reaches a certain value such as $H_c$, the energy supply has increased to such an extent that any further increase is negligible compared with the quantity itself, i.e., $\Delta R_m \ll R_m$. The energy-speed radius can thus be regarded as constant, and (4.20) then shows that $V_1$ decreases with increasing mountain height $H_1$ (or $\lambda_0$) while $U_0$ reaches an asymptotic value (Fig. 4.4b).

Furthermore, the response of the surface wind to mechanical forcing is linear in (4.19) when $h$ is small and nonlinear in (4.20) when $h$ is large. The validity of the linear theory is therefore justified by the inequality

$$h < H_c.$$  \hfill (4.21)

Here $H_c$ is the critical mountain height given by

$$H_c = \frac{8}{3} \sqrt{2} \alpha (\sin \lambda_0)^{-1}. \quad (4.22)$$

If the integration domain for Eqs. (4.4) and (4.12) is a spherical surface rather than a plane, similar results can be obtained except that the denominator in (4.22) is replaced by $\pi$. Equations (4.22) and (4.21) show that the applicability of the linear response depends on the frictional coefficient, and the phase difference between the surface pressure perturbation and the orography. Taking $\alpha = (K/2f_0)^{1/2} = 230$ m and $\lambda_0$ from $\pi/2$ to $\pi/6$, $H_c$ varies between 870 m and 1.75 km, in accord with our previous discussion summarized in Table 4.1 and shown in Fig. 4.2.

Since the theory of nonlinear response of the atmosphere to mechanical forcing is based on the universal relations of energy [Eqs. (4.1) and (4.2)] and the requirement of angular momentum balance (4.10), independent of any general circulation model, we can therefore infer that this nonlinear theory represented by (4.15)–(4.22) is applicable in general, and that the picture given by Fig. 4.2 is a representative result.

5. Thermal–mechanical forcing

That the monthly normals are quasi-stationary implies that these components are excited by geographically fixed sources. As discussed in Sections 3 and 4, large-scale mountains may be one of these sources. However, when the annual variation of the monthly normals at lower levels are investigated, it

**Fig. 4.4.** Nonlinear response of surface wind to the mountain height. (a) When the mountain is shallow, both $V_1$ and $U_0$ increase as the mountain height, represented by $\tan \theta_1$, increases. (b) When the mountain is large and the height is increased, $V_1$ decreases while $U_0$ increases to approach an asymptotic value $R_m$. 


Fig. 5.1. Diabatic heating (K day$^{-1}$) applied to the model atmosphere in the thermal forcing and thermal-mechanical forcing experiments.

is clearly seen that these "geographically fixed sources" are no longer confined to the mountain ranges alone. For example, surface anticyclones occur over the continents and cyclones over the oceans in winter, and vice versa in summer; the persistence of both monthly mean flow patterns and the departures from these normals breaks down after the equinoxes (Nami, 1952), or in June and October in Asia (Yeh and Guo, 1958). In addition, the distribution of the mountain torque in subtropical latitudes, where most of the Tibetan plateau is located, changes sign from winter to summer, in strong contrast to its constancy in other latitude belts (Yeh and Zhu, 1958). All these observations imply the relative importance of land-ocean thermal contrasts to lower atmospheric systems. Even during the persistent periods we can see the influence of diabatic heating on the atmosphere. A particular example is the structure of the large-scale systems over the eastern part of the Eurasian continent. Newton (1971a) showed that mountain torque due to the Tibetan plateau changes sign at a height of ~3 km in winter. Shon and Li (see Yeh and Guo, 1979) calculated the temperature difference between the eastern and western sides of the plateau, showing that the vertical sign change occurs at similar heights. Newton's discovery may thus be attributed to the fact that the Siberian High at subtropical latitudes is a shallow system, and it also implies that mountain torque results from the thermal, as well as the mechanical effect.

Smagorinsky (1953) and Döös (1962) showed that heat sources and sinks of planetary dimensions can produce quasi-stationary perturbations. This theory is investigated in this Section by experiments with purely thermal forcing. Putting

$$\begin{align*}
SA_2^0 &= -22 \text{K/Ts} , \\
SA_2^1 &= -10 \text{K/Ts} , \\
SB_2^0 &= -15 \text{K/Ts} , \\
SA_3^2 &= 37 \text{K/Ts} , \\
SA_3^2 &= 5 \text{K/Ts} \\
\end{align*}$$

(5.1)

for the reference temperature field $\tilde{T}$, we represent the diabatic heating field $-\gamma(T - \tilde{T})$ in our model as shown in Fig. 5.1, with atmospheric cooling over two continents and warming over two oceans. Figure 5.2 shows the surface pressure field $P_0$ which is excited. Two continental anticyclones are located about 15–25° to the east of the heat sinks, while two oceanic lows occur a similar distance to the east of the heat sources, very similar to the results of Smagorinsky and Döös.

Based on a nonlinear balance equation model, Ashe (1979) showed that none of the major asymmetric flow features can be attributed solely to a thermal or orographic origin. In order to study the combination of mechanical and thermal effects, we create a thermal-mechanical (TM) model by adding the asymmetric diabatic forcing given by (5.1) and shown in Fig. 5.1 to the previous mechanical forcing with $hA_3^0 = -500 \text{m/hs}$ and $hB_3^1 = 250 \text{m/hs}$ (Fig. 3.2). The coincidence of the two mountain peaks with the two cooling centers represents the winter situation in the Northern Hemisphere. The surface pressure field excited by this TM forcing is shown in

Fig. 5.2. Response of sea-level pressure (mb) to the thermal forcing shown in Fig. 5.1.
Fig. 5.3, and we notice that it is more or less similar to that resulting from the purely thermal forcing presented in Fig. 5.2. However, orography decreases the perturbation amplitude from 45 to 40 mb, but essentially increases its latitudinal gradient from 2 to 13 mb. This is in agreement with the conclusion we obtained earlier from the nonlinear theory developed in the surface wind space \( \{U_1, V_1, U_0\} \). By contrast, Fig. 5.3 is totally different from that due to purely mechanical forcing as shown in Fig. 4.1a. The two cyclones on the eastern sides of the mountains in the case of pure \( M \) forcing have moved to the west of the mountains in the TM-forcing situation.

It is natural to think that since the effect of mechanical forcing on the surface systems is small compared to the effect of thermal forcing, the combined effect must be similar to that of thermal forcing alone. In order to test this reasoning, we construct the sea-level pressure field by adding Fig. 4.1a to Fig. 5.2 algebraically, and show the result in Fig. 5.4. This may be referred to as a linear-combination thermal-mechanical (T + M) forcing. Compared with Fig. 5.2, we see that the surface pressure field resulting from the (T + M)-forcing is basically the same as that due to purely mechanical forcing: the effect of mechanical forcing on the surface systems is so small that it is, in fact, negligible in the case of (T + M) forcing. On the other hand, when Fig. 5.3 is compared with Fig. 5.2, we find that the effect of mechanical forcing on the location and intensity of the sea-level systems becomes much stronger in the case of the nonlinear TM forcing. For example, the addition of the perturbation pressure \( \Delta P_0^\theta \) due to the purely mechanically forced case to that due to the purely thermally forced case only reduces \( \Delta P_0^\theta \) by 1 mb to 44 mb. The much stronger decrease of \( \Delta P_0^\theta \) (5 mb) observed from the TM-forcing (Fig. 5.3) is therefore due not only to the out-of-phase structure of the sea-level pressure fields in T and M forcing, but also, and more importantly, to the nonlinear response of the atmosphere to mechanical forcing.

The most striking difference between the (T + M) and TM forcing occurs with the pressure difference between the pole and the equator, \( \Delta P_0 \). By contrast to the 11 mb decrease in the nonlinear case, it only decreases from 2 mb in the TM case to zero in the (T + M) case. It is clear that the nonlinear effect is much more important than the linear effect for the formation of the sea-level pressure pattern with the combined of T and M forcing. This can also be understood from the nonlinear theory we developed earlier in the surface-wind space \( \{U_1, V_1, U_0\} \).

It is significant that, without mountains and without vorticity forcing, the zonal mean zonal surface wind is very weak in purely thermal forcing. When a mountain is put into the model, a surface zonal wind is excited, as discussed in Section 4. In a steady state, a given supply of mechanical energy must correspond to the same frictional dissipation of energy. Hence, the surface zonal wind is produced at the expense of a decrease in the perturbation wind. It seems that although large-scale systems at low levels result mainly from thermal forcing, their locations and intensities are to some extent affected by mechanical forcing.
6. The balance of angular momentum

Angular momentum transfer by planetary waves in the case of purely mechanical forcing is so small that its role in angular momentum balance is negligible. The surface frictional torque is therefore balanced mainly by the mountain torque in every latitude belt, i.e.,

$$f_0 p_0 v \Phi a \cos \varphi = f_0 p_0 \alpha \bar{u}_0 \cos \varphi.$$  \hspace{1cm} (6.1)

Figure 6.1 gives the balance of angular momentum with $HA_0^2 = -2000$ m$/Hs$ and $HB_0^1 = 1000$ m$/Hs$ (referred to Fig. 4.1b). At first glance, the signs of the frictional and mountain torque are contrary to what is observed in winter, i.e., the net frictional and mountain torques over the Northern Hemisphere are negative and positive, respectively (Newton, 1971b; Oort and Bowman, 1974). More interestingly, under the deep westerlies the zonal surface wind is reversed to easterly along every latitude circle. The reason is simply the requirement for angular momentum balance: in a steady state, the loss of angular momentum from the atmosphere to the earth due to the westward mountain torque should be regained from the earth through the surface easterlies. Of course, this is not the process in the real atmosphere. The assumption that mechanical forcing is dominant in the atmosphere cannot be proved correct as far as atmospheric angular momentum balance is concerned.

The latitudinal distributions of surface frictional torque $T_f$, mountain torque $T_M$, and zonal mean zonal surface wind $\bar{u}_0$ in the TM case with $HA_0^2 = -2000$ m$/Hs$ and $HB_0^1 = 1000$ m$/Hs$, are represented in Fig. 6.2. Compared with Fig. 6.1, every quantity in the TM case possesses the opposite sign to its counterpart in the M case because the surface pressure system is maintained mainly by the thermal forcing. Surface anticyclones are produced on the eastern sides of mountains, resulting in positive mountain torque. The transfer of angular momentum by planetary waves is very weak; in order to satisfy the balance requirement, westerly momentum must be removed by friction, resulting in westerly surface winds at every latitude.

The reversal of the direction of the zonal mean zonal velocity near the surface in purely mechanical forcing now disappears. The sign of the mountain torque is also consistent with that of the net mountain torque observed in the Northern Hemisphere winter. There is, therefore, no doubt that TM forcing produces more reasonable and realistic results than either purely thermal or mechanical forcing, as far as the balance of angular momentum of the atmosphere is concerned, even though the overall balance of angular momentum of the real atmosphere must depend on the combined effects of thermal and mechanical forcing, and vorticity forcing resulting from the transfer properties of transient eddies (Wu, 1983). Our experiments also make it clear that although the mountain torque owes its existence to the unevenness of the earth's surface, its sign depends critically upon the relative positions of heat sources and mountains. In other words, the observed mountain torque depends critically on the relative positions of mechanical and thermal forcing along latitude circles. The statement made by some authors that mountain torque is produced by the lee-side trough seems misleading. Since mountains are fixed geographically, the annual variation of mountain torque in subtropical areas (Yeh and Zhu, 1958) must be the result of the annual variation of thermal forcing at those latitudes.
7. Conclusions

1) By exerting drag on the atmosphere, mountains react to the basic current impinging on them. This reaction causes a reduction in the zonal mean angular momentum of the atmosphere. At the same time, part of the zonal mean kinetic energy is converted to eddy kinetic energy, resulting in a stationary mountain wave. In the frictional case, the mechanically forced surface anticyclone is located on the western side of the mountain crest.

In the baroclinic case, the mountain wave is coupled with a thermal wave which lags to the west; sensible heat is then transferred poleward. Consequently, the wave develops during its upward propagation by drawing on the zonal mean energy. The work done on the air by the perturbation pressure therefore increases with height.

2) For the response of the atmosphere to mechanical forcing, there exists a critical mountain height \( h_c \approx 8/3(\sin \lambda_0)^{-1} \), which is a function of the frictional coefficient as well as the phase difference between the mountain and the surface pressure field.

If, and only if, the mountain height is less than this critical value, can the deflection effect of the mountain be neglected and the response be regarded as approximately linear. This critical mountain height is only about 1 km. Thus most of the atmospheric response to large scale mountains must be nonlinear.

In the nonlinear case, as the mountain height is increased the deflection effect becomes increasingly important. Therefore, although in the upper atmosphere the mountain wave is intensified, on the surface the pressure perturbation decreases while the zonal surface easterlies become dominant and approach an asymptotic value.

3) Purely mechanical forcing produces negative mountain torque and results in zonal mean surface easterlies over most latitudes in both the barotropic and baroclinic cases. This is inconsistent with the fact that mountain torque is balanced to a major extent by factors other than the surface frictional torque. Purely mechanical forcing cannot, therefore, be the case in the real atmosphere.

4) In the extratropical atmosphere, large-scale diabatic heating must be balanced mainly by horizontal advection of entropy. Therefore, a heat sink results in a downstream surface cold anticyclone, and a heat source in a downstream surface warm cyclone. Thermal forcing is more important than mechanical forcing in the maintenance of the surface pressure field.

5) Although the mountain torque owes its existence to the unevenness of the earth's surface, its sign and intensity depend critically upon the relative locations of the mechanical and thermal forcing. The Tibetan plateau plays the roles of a heat sink in winter and a source in summer. Hence, on its eastern side an anticyclone occurs in the winter and a cyclone in summer. This produces different mountain torques in different seasons, and accounts for the prominent annual variation of mountain torque in subtropical latitudes.

6) The combination of the effects of mechanical and thermal forcing is also nonlinear. The inclusion of orography in a thermally forced atmosphere reduces the perturbation of surface pressure, but essentially increases its latitudinal gradient. Also, the increase in mountain height results in a westward shift of the surface anticyclone. Despite the fact that the formation of surface pressure systems is mainly a result of thermal forcing, orography affects to some extent their intensities and locations.

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APPENDIX A

Numerical Solutions of the Model

Upon substituting (2.8) and (2.12) into (2.7), the prognostic equations turn out to be (2.13), where

\[
G_1 = -\frac{1}{2} \left[ J(\psi_M, \nabla^2 \psi - \mathbf{v}) + J(\psi, \nabla^2 \psi_s) \right] - J(\psi_M - \psi_s, \eta) - 2 \frac{\partial \psi_M}{\partial \lambda} - R_d(\nabla^2 \psi_M - 2 \nabla^2 \psi_s) \\
+ \frac{1}{2} \nabla \cdot \left[ K \nabla(\nabla^2 \psi_M + \nabla^2 \psi_s + 2f - \mu^2 \psi_s) + K \nabla(\nabla^2 \psi_M - \nabla^2 \psi_s + 2f + \mu^2 \psi_s) \right],
\]

\[
G_2 = -\frac{1}{2} \left\{ J(\psi_M, (\nabla^2 - \mu^2) \psi_s) + J(\psi, \nabla^2 \psi_M) \right\} + J(\psi_M - \psi_s, \eta) - 2 \frac{\partial \psi_s}{\partial \lambda} + R_d(\nabla^2 \psi_M - 2 \nabla^2 \psi_s) \\
+ \frac{1}{2} \nabla \cdot \left[ K \nabla(\nabla^2 \psi_M + \nabla^2 \psi_s + 2f - \mu^2 \psi_s) - K \nabla(\nabla^2 \psi_M - \nabla^2 \psi_s + 2f + \mu^2 \psi_s) \right] + \mu^2 \gamma(\psi_s - \psi_M).
\]
On substituting (2.14) into (2.13) and integrating the resulting equations with respect to \( \lambda \) and \( x \) from 0 to \( 2\pi \), and from 0 to 1, respectively, we obtain the 20 "tendency" equations as in (2.19), and the constraint (2.20). Here,

\[
F_1 = -\frac{1}{2C_n} \sum C_n \left\{ [(A_m^m B_n^m + B_m^m A_n^m + a_m^m a_n^m + b_m^m a_m^m) (m_2 I_{n^m}^m - m_3 I_{n^m}^m)]_{m=m+2m_3} \\
+ [(B_m^m A_n^m - A_m^m B_n^m + b_m^m a_m^m - a_m^m a_n^m) (m_2 I_{n^m}^m + m_3 I_{n^m}^m)]_{m=m+2m_3} \right\} \\
- \frac{1}{C_n} \sum \right\{ [(DA_m^m h B_n^m + DB_m^m h A_n^m) (m_2 I_{n^m}^m - m_3 I_{n^m}^m)]_{m=m+2m_3} \\
+ [(DB_m^m h A_n^m - DA_m^m h B_n^m) (m_2 I_{n^m}^m + m_3 I_{n^m}^m)]_{m=m+2m_3} \right\} \\
- \frac{1}{C_n} 2mB_m^m - R_d (A_n^m - 2a_n^m) + \frac{2}{C_n} \sum \alpha_i^m R_i^m + \frac{8}{C_n} k_\nu Q_n^0_{m=0,m=0},
\]

\[
F_2 = -\frac{1}{2C_n} \sum C_n \left\{ [(B_m^m A_n^m - A_m^m B_n^m + b_m^m a_m^m - a_m^m a_n^m) (m_2 I_{n^m}^m - m_3 I_{n^m}^m)]_{m=m+2m_3} \\
+ \text{sgn}(m_3 - m_2) [(A_m^m A_n^m + B_m^m B_n^m + a_m^m a_n^m + b_m^m b_n^m) (m_2 I_{n^m}^m + m_3 I_{n^m}^m)]_{m=m+2m_3} \right\} \\
- \frac{1}{C_n} \sum \right\{ [(DA_m^m h B_n^m - DA_m^m h A_n^m) (m_2 I_{n^m}^m - m_3 I_{n^m}^m)]_{m=m+2m_3} \\
+ \text{sgn}(m_3 - m_2) [(DB_m^m h A_n^m - DA_m^m h B_n^m) (m_2 I_{n^m}^m + m_3 I_{n^m}^m)]_{m=m+2m_3} \right\} \\
- \frac{2}{C_n} \sum \beta_i^m R_i^m,
\]

\[
F_3 = -\frac{1}{2G_n} \sum \left\{ [(G_m^m (A_n^m b_{n^m}^m + B_m^m a_{n^m}^m)) + C_n (a_m^m B_n^m + b_m^m A_n^m)] (m_2 I_{n^m}^m - m_3 I_{n^m}^m)]_{m=m+2m_3} \\
+ [G_n (B_m^m a_m^m - A_m^m B_n^m) + C_n (b_m^m A_n^m - a_m^m B_n^m)] (m_2 I_{n^m}^m + m_3 I_{n^m}^m)]_{m=m+2m_3} \right\} \\
+ \frac{1}{G_n} \sum \\right\{ [(DA_m^m h B_n^m + DB_m^m h A_n^m) (m_2 I_{n^m}^m - m_3 I_{n^m}^m)]_{m=m+2m_3} \\
+ [(DB_m^m h A_n^m - DA_m^m h B_n^m) (m_2 I_{n^m}^m + m_3 I_{n^m}^m)]_{m=m+2m_3} \right\} \\
- \frac{2}{G_n} \sum \frac{ma_m^m + R_d C_n}{G_n} (A_n^m - 2a_n^m) + \gamma^* \frac{k_\nu}{G_n} (a_m^m - S A_m^m) + \frac{2}{G_n} \sum S \alpha_i^m R_i^m + \frac{8}{G_n} k_\nu Q_n^0_{m=0,m=0},
\]

\[
F_4 = -\frac{1}{2G_n} \sum \left\{ [(G_m^m (B_n^m b_{n^m}^m - A_m^m a_{n^m}^m)) + C_n (b_m^m B_n^m - a_m^m A_n^m)] (m_2 I_{n^m}^m - m_3 I_{n^m}^m)]_{m=m+2m_3} \\
+ \text{sgn}(m_3 - m_2) [(G_m^m (A_n^m a_{n^m}^m + B_m^m b_{n^m}^m)) + C_n (a_m^m A_n^m + b_m^m B_n^m)] (m_2 I_{n^m}^m + m_3 I_{n^m}^m)]_{m=m+2m_3} \right\} \\
+ \frac{1}{G_n} \sum \\right\{ [(DB_m^m h B_n^m - DA_m^m h A_n^m) (m_2 I_{n^m}^m - m_3 I_{n^m}^m)]_{m=m+2m_3} \\
+ \text{sgn}(m_3 - m_2) [(DA_m^m h A_n^m + DB_m^m h B_n^m) (m_2 I_{n^m}^m + m_3 I_{n^m}^m)]_{m=m+2m_3} \right\} \\
+ \frac{2}{G_n} \sum \frac{m a_m^m + R_d C_n}{G_n} (B_n^m - 2b_n^m) + \gamma^* \frac{k_\nu}{G_n} (b_m^m - S B_m^m) + \frac{2}{G_n} \sum S \beta_i^m R_i^m,
\]

where

\[
d_1 = \sum_{n=2,4} \frac{F_n}{C_n} \left[ 4Q_n^0 + \sum i (C_i A_i - G A_i) R_i^0 \right],
\]

\[
d_3 = \sum_{n=2,4} \frac{F_n}{C_n} \left[ 4Q_n^0 + \sum i (C_i A_i + G A_i) R_i^0 \right],
\]

\[
d_4 = \frac{2}{\rho_0 Ha^2} \int_0^1 S(x) dx,
\]

\[
I_{n^m}^{m_3} = \int_0^1 P_m^m P_{n^m}^m d \frac{dx}{(P_{m_3}^m)} dx,
\]

\[
\text{sgn}(m_3 - m_2) = \begin{cases} 1 & m_3 > m_2 \\ 0 & m_3 = m_2 \\ -1 & m_3 < m_2 \end{cases}
\]
\[ Q_m^0 = \left\{ \begin{array}{ll}
0 & m = 0 \\
\int_0^1 P_m^0 \left[ (1 - x^2) \frac{dP_m^3}{dx} - 2xP_m^3 \right] dx & m \neq 0,
\end{array} \right. \\
R_m^0 = \int_0^1 P_m^0 \left[ C_l P_m^3 P_l^m + \frac{dP_m}{dx} \right] \left[ (1 - x^2) \frac{dP_3^3}{dx} \right] dx,
F_m = \int_0^1 (1 - x^2) \frac{dP_0^0}{dx} dx,
D_A^m = A_m^m - a_n^m,
D_B^m = B_m^m - b_n^m,
\sum^4 = \sum_{n_2} \sum_{m_2} \sum_{n_3} \sum_{m_3},
\alpha_n^m = C_n k_M A_n^m + G_n k_a b_n^m,
\beta_n^m = C_n k_M B_n^m + G_n k_b b_n^m,
\gamma_n^m = C_n k_M A_n^m + G_n k_a b_n^m,
\delta_n^m = C_n k_M B_n^m + G_n k_b b_n^m,
k_s = (k_3 - k_1)/2,
k_M = (k_3 + k_1)/2,
C_n = -n(n + 1),
G_n = C_n - \mu^2,
\]

and the subscript equalities represent conditions for the existence of the corresponding terms.

Now, (2.19) and (2.20) form a closed set with 21 equations and 21 variables.

APPENDIX B
Stability of the Time Integration of the Parameterized Terms

In order to integrate the potential vorticity equation

\[ \frac{\partial q}{\partial t} = \frac{\partial}{\partial x} \left( k \frac{\partial q}{\partial x} \right) + \text{other terms}, \tag{B1} \]

where

\[ K = kP_3^3 = k2.56174(x - x^3) = A(x - x^3), \tag{B2} \]

we divide the interval \((0, 1)\) into \(N\) equal segments with the increment \(h = 1/N\) and let \(\tau\) be the temporal increment. At \(t = t_m = mh\) and \(x = x_j = jh\), the time-forward difference form of (B1) becomes

\[ q_j^{n+1} - q_j^n = \frac{\Delta t}{2h^2} \{ (c_j + d_j)(q_j^{n+1} - q_j^n) \}
- (c_j - d_j)(q_j^n - q_{j-1}^n), \]

where

\[ c_j = 2jh - (2j^3 - 3j)h^3, \]
\[ d_j = h - (3j^2 + 1)h^3 \]. \tag{B3}

Let the calculation error in \(q\) be

\[ e_j^n = \lambda^n e_j^0 \tag{B4} \]

and assume the initial error

\[ e_j^0 = \exp(ikx_j), \tag{B5} \]

then the error equation becomes

\[ e_j^{n+1} - e_j^n = b\{(c_j + d_j)(e_j^{n+1} - e_j^n) \}
- (c_j - d_j)(e_j^n - e_{j-1}^n) \}
\]

where

\[ b = A\tau/2h^2. \tag{B6} \]

On substituting (B4) and (B5) into the error equation, we obtain

\[ \lambda = 1 - \frac{\tau A}{h^2} \{2c_j \sin^2(kh/2) - i d_j \sin kh \}. \tag{B7} \]

Since \(kh \ll 1\),

\[ \lambda \approx 1 - b\{c_j k^2 h^2 - i d_j kh\}
= (1 - bc_j k^2 h^2) + i 2bd_j kh. \]

Thus

\[ |\lambda|^2 = 1 - 2bc_j k^2 h^2 + b^2 c_j^2 (kh)^4 + 4b^2 d_j^2 (kh)^2. \]

By choosing \(b \ll 1\), then since

\[ 1 > c_j \gg d_j, \]
\[ kh \ll 1, \]
so

\[ |\lambda|^2 \approx 1 - 2bc_j (kh)^2. \]

Therefore the stability requirement becomes

\[ b \leq \frac{1}{2c_j (kh)^2} \quad \text{for all } j \tag{B8} \]

or

\[ \tau \leq \frac{1}{A c_{\text{max}} k^2}. \tag{B9} \]

From (B3), \(c_j\) reaches a maximum if

\[ j = [-3 + (9 + 48N^2)^{1/2}] / 12. \]

Hence for \(N = 18\), \(c_{\text{max}} = 0.763\). Let

\[ k = 2\pi/(6 \times 10^6 \text{ m}), \quad A = 2.56174 \times 4 \times 10^6 \text{ m}^2 \text{ s}^{-1}, \]

then when

\[ \tau < 116633 \text{ s} \approx 1.3 \text{ day}, \]

the time integration is stable.

For \(K = \text{constant}\), (B1) becomes a canonical diffusion equation. It can be shown that the stability requirement now becomes

\[ \tau < 4/Kk^2. \tag{B10} \]

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