Computational stability of the forced dissipative nonlinear atmospheric equations

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Abstract A new concept of computational quasi-stability (CQS) is introduced to study the computational stability (CS) of the forced dissipative nonlinear (FDN) evolution equations. Based on the concept, the CQS criterion of difference scheme of FDN atmospheric equations is obtained. So it provides the theoretical basis for designing the computational stable difference scheme of FDN atmospheric equations.

Keywords: computational stability (CS), computational quasi-stability (CQS), operator equation, difference scheme, forced dissipative nonlinear (FDN) equations.

THE computational stable numerical scheme must be designed if we solve numerically the dynamic equations of the nonlinear atmosphere or ocean. Zeng and Ji *et al*.[1–5] have carried out lots of work for the adiabatic or non-dissipative nonlinear evolution equations. For the forced dissipative nonlinear (FDN) equations, however, its computational stability (CS) analysis has not yet been dealt with so far because of its inherent difficulties. In practice, it is very important to study the CS of FDN evolution equations. This note presents a new concept, i.e. the computational quasi-stability (CQS), and based on it, the CS of FDN evolution equations is investigated.

1 Basic description and the concept of CQS

The complete FDN atmospheric equations can be transformed into an equivalent operator equation in Hilbert space as follows[6]:

$$\frac{\partial \varphi}{\partial t} + N(\varphi)\varphi + L(\varphi)\varphi = \xi, \tag{1}$$

$$\varphi|_{t=0} = \varphi_0, \tag{2}$$

where $N(\varphi)$ is an anti-adjoint operator, $L(\varphi)$ a self-adjoint operator, i.e.

$$N(\varphi) = -N^*(\varphi), \quad (N(\varphi)\varphi, \varphi) = 0, \tag{3}$$

$$L(\varphi) = L^*(\varphi), \quad (L(\varphi)\varphi, \varphi) \geqslant 0. \tag{4}$$

The equal in (4) is true if and only if $\|\varphi\| = 0$.

Here we use difference methods to solve the approximate solution of (1) and (2). Let τ be the time stepsize, h the space stepsize, setting the mesh $(mh, n\tau)$, and let φ^n be the value of φ at the time t_n. We define the inner product of the mesh function as

$$(\varphi, \tilde{\varphi}) = \sum_m \varphi_m \tilde{\varphi}_m \Delta_m, \tag{5}$$

where φ, $\tilde{\varphi}$ are two arbitrary abstract functions (They are usually vector functions), φ_m, $\tilde{\varphi}_m$, Δ_m are the values of φ, $\tilde{\varphi}$ and the unit volume at mth mesh point respectively. The norm of the function φ is defined as

$$\|\varphi\| = (\varphi, \varphi)^{1/2}. \tag{6}$$

Hereafter let $N(\varphi^*)$ and $L(\varphi^*)$ be the general discrete forms of $N(\varphi)$ and $L(\varphi)$ respectively. The antisymmetric operator in Hilbert space corresponds to the antisymmetric matrix in R^n, and the symmetric positive operator the symmetric positive matrix, so $N(\varphi^*)$ is an anti-symmetric matrix and $L(\varphi^*)$ a symmetric positive matrix.

For eq. (1), it is quite difficult to discuss the CS for its difference scheme. It is usually studied only when $\xi \equiv 0$[1—4]. In the case of $\xi \neq 0$, there is not any result at present. Hence, it is necessary to soften the terms and introduce the concept of CQS as follows.

Definition 1. When the time stepsize τ is sufficiently small, if the numerical solution computed by the difference method satisfies

$$\|\varphi^{n+1}\| \leqslant \|\varphi^n\| + \tau c, \tag{7}$$

where c is a constant depending on $\|\xi\|$, then the difference scheme is called the computational quasi-stability (CQS).

It is obvious that the CQS is a necessary condition of CS. The difference scheme must be computationally instable if it does not satisfy the condition of CQS. The CQS is just the CS for $\xi \equiv 0$ or $\tau \to 0$.

2 Main results

The general difference scheme of eq. (1) is given as follows:

$$\frac{\varphi^{n+1} - \varphi^n}{\tau} + N(\varphi^*)[a\varphi^{n+1} + (1-a)\varphi^n] + L(\varphi^*)[a\varphi^{n+1} + (1-a)\varphi^n] = \xi, \; 0 \leqslant a \leqslant 1. \tag{8}$$

Theorem 1. The difference scheme (5) of eq. (1) is computationally quasi-stable for $0 \leqslant a \leqslant 1/2$.

Proof. Making inner product with $a\varphi^{n+1} + (1-a)\varphi^n$ for both sides of (8), using the properties of $N(\varphi^*)$ and $L(\varphi^*)$, we have

$$\left(\frac{\varphi^{n+1} - \varphi^n}{\tau}, a\varphi^{n+1} + (1-a)\varphi^n\right) \leqslant (\xi, a\varphi^{n+1} + (1-a)\varphi^n),$$

i.e.

$$a\|\varphi^{n+1}\|^2 \leqslant (1-a)\|\varphi^n\|^2 + (2a-1)(\varphi^n, \varphi^{n+1}) + \tau(\xi, a\varphi^{n+1} + (1-a)\varphi^n)$$
$$\leqslant (1-a)\|\varphi^n\|^2 + \frac{2a-1}{2}\|\varphi^n\|^2 + \frac{2a-1}{2}\|\varphi^{n+1}\|^2$$

$$+ a\tau \| \xi \| \| \varphi^{n+1} \| + (1-a)\tau \| \xi \| \| \varphi^n \| .$$

So

$$\| \varphi^{n+1} \|^2 - 2a\tau \| \varphi^{n+1} \| \| \xi \| \leqslant \| \varphi^n \|^2 + 2(1-a)\tau \| \varphi^n \| \| \xi \| .$$

Therefore for $1/2 \leqslant a \leqslant 1$ one gets

$$\begin{aligned} & \| \varphi^{n+1} \|^2 - 2a\tau \| \varphi^{n+1} \| \| \xi \| + a^2\tau^2 \| \xi \|^2 \\ \leqslant & \| \varphi^n \|^2 + 2(1-a)\tau \| \varphi^n \| \| \xi \| + a^2\tau^2 \| \xi \|^2 \\ \leqslant & \| \varphi^n \|^2 + 2a\tau \| \varphi^n \| \| \xi \| + a^2\tau^2 \| \xi \|^2, \end{aligned}$$

namely,

$$\begin{aligned} \| \varphi^{n+1} \| & \leqslant \| \varphi^n \| + 2a\tau \| \xi \| \\ & \leqslant \| \varphi^n \| + \tau c, \end{aligned}$$

where $c = 2 \| \xi \|$. So (8) is computationally quasi-stable for $1/2 \leqslant a \leqslant 1$. The proof is completed.

Especially, for $\xi \equiv 0$, $c = 0$. In this case, one has

$$\| \varphi^{n+1} \| \leqslant \| \varphi^n \| ,$$

as a result scheme (8) is computationally stable.

For the nonlinear equations, the CS criterion of their linearized equations has also reference values. Thus we make an additional analysis for the linearized equation of (1). The linearized equation of (1) is

$$\frac{\partial \varphi}{\partial t} + N(\overline{\varphi})\varphi + L(\overline{\varphi})\varphi = \xi, \tag{9}$$

Its difference scheme corresponds to the form

$$\frac{\varphi^{n+1} - \varphi^n}{\tau} + (N(\overline{\varphi}) + L(\overline{\varphi}))(a\varphi^{n+1} + (1-a)\varphi^n) = \xi. \tag{10}$$

Theorem 2. The difference scheme (10) of Eq. (9) is computationally stable for $1/2 \leqslant a \leqslant 1$.

Proof. Let φ_1, φ_2 be two arbitrary solutions, and

$$\varepsilon^i = \varphi_1^i - \varphi_2^i, \quad i = 1, 2, \cdots, n,$$

and thus

$$\frac{\varepsilon^{n+1} - \varepsilon^n}{\tau} + (N(\overline{\varphi}) + L(\overline{\varphi}))(a\varepsilon^{n+1} + (1-a)\varepsilon^n) = 0.$$

For $1/2 \leqslant a \leqslant 1$, it follows that

$$| \varepsilon^{n+1} | \leqslant | \varepsilon^0 | ,$$

and therefore the scheme is computationally stable. The proof is completed.

According to the above results, when $1/2 \leqslant a \leqslant 1$, scheme (8) is computationally stable for the linearized equation (with the external focing) or the adiabatic case (with the nonlinear terms) and is computationally quasi-stable for the FDN equation. Therefore , the nonlinear interaction and the effect of the external focing are the main factors which cause the error propagation for CQS scheme while disregarding round-off error.

3 Conclusion

Because of the complexity of CS of FDN evolution equations, this note presents a new concept, CQS, to study it, and gives the CQS criterion of the difference scheme of the FDN atmospheric equations. The criterion is also the CS criterion of the linearized equations or the adiabatic case, and the CQS is a necessary condition of the CS, so the computationally stable scheme must be in the criterion. The results furnish the necessary theoretical basis for designing the computationally stable difference scheme of the FDN atmospheric equations.

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